A tropical isoperimetric inequality

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Abstract. We introduce tropical analogues of the notion of volume of polytopes, leading to a tropical version of the (discrete) classical isoperimetric inequality. The planar case is elementary, but a higher-dimensional generalization leads to an interesting class of ordinary convex polytopes, characterizing the equality case in the isoperimetric inequality. This study is motivated by open complexity questions concerning linear optimization and its tropical analogs.

Keywords: Tropical geometry, polytopes, log-limit sets, volume, idempotent measures

1 Introduction

The classical isoperimetric inequality states that the bounded planar region with given perimeter which maximizes the area is the circular disk. Its discrete version, from which a proof of the smooth result can be derived, says that the triangle with fixed perimeter which maximizes the area is equilateral; see [4] for a nice survey. A minor variation of the same problem asks to maximize the area for fixed diameter (instead of fixed perimeter), and it has the same result. The tropical analog is a statement about the tropical convex hull of three points in the plane. Our main contribution is a generalization of that result to arbitrary dimensions.

This work is motivated by research on delicate complexity issues related with classical linear programming. In [3] a family of linear programs was constructed which exhibits central paths with unusually large total curvature. These linear programs provide counter-examples to a "continuous analog of the Hirsch conjecture" by Deza, Terlaky and Zinchenko [11]. The key idea in [3] was to obtain a lower bound for the total curvature of the central path by means of a piecewise-linear curve which can be associated

with the tropicalization of linear program. In this way discrete notions of curvature, which make sense from a tropical geometry perspective, give rise to non-trivial metric estimates for classical curves. This lead us to further investigate aspects of tropical geometry in the spirit of discrete differential geometry [5]. In particular, we are interested in tropical versions of the isoperimetric inequalities.

Tropical linear algebra is concerned with $(\max, +)$- or $(\min, +)$-analogs of classical linear algebra. A *tropical polyhedral cone* is a set of points in $\mathbb{R}^d$ which are tropical linear combinations of finitely many generators. Its image under the projection modulo the all-ones vector is a *tropical polytope*; see [8, 21] and the references there for an overview of the theory. The search for a tropical analogue of volume lead us to propose a new notion which captures the metric intuition of tropical polytopes well enough. Our main result shows that the tropical simplices which maximize this tropical volume for fixed tropical diameter are convex in the ordinary sense, i.e., these are the *polytropes* studied in [17]. The polytropes form the combinatorial building blocks of tropical convexity [21, §5.2]. In combinatorial optimization they arise naturally in the study of shortest path algorithms [23, §8.3], [24], [18]. Furthermore, they are isomorphic to tropical eigenspaces (see e.g. [7, Chapter 4]), play a role in the theory of semigroups [16] and occur in statistical ranking [25]. It is known that, up to symmetry, there is precisely one generic combinatorial type of polytropes in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ [17]. The generic polytropes in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ were classified in [15] (see also [26]); there are precisely six types. However, it turns out that, at least in these dimensions, only one generic type maximizes the tropical volume for fixed tropical diameter.

A second approach to obtain a tropical analogue of volume is to employ the "dequantization" method [19], thinking of a tropical polytope as a log-limit of a family of classical polytopes, and defining the limit of the normalized volumes of these polytopes. We shall see that the volume obtained in this way also has several good properties. For instance, it turns out to be an idempotent measure on the space of generic tropical polytopes and easy to compute. However, this yields a degenerate isoperimetric inequality.

2   Tropical distance and volume

For two points $v, w \in \mathbb{R}^d$ the *tropical distance* is the number

$$\begin{aligned} \mathrm{tdist}(v, w) \ &:= \ \max\big\{ (v_i - w_i) \mid i \in [d] \big\} - \min\big\{ (v_i - w_i) \mid i \in [d] \big\} \\ &= \ \max_{i,j \in [d]} \big| v_i - w_i + w_j - v_j \big| \ . \end{aligned}$$

This number was shown in [8] to play the role of the Euclidean distance in the tropical setting. It is a special instance of Hilbert's projective metric. We have

$$
\begin{aligned}
\text{tdist}(\mathbf{1} + v, w) &= \text{tdist}(v, w) \quad \text{and} \\
\text{tdist}(u + v, u + w) &= \text{tdist}(v, w) \quad \text{for all } u \in \mathbb{R}^d \ .
\end{aligned}
\tag{2.1}
$$

In particular, this entails that tdist induces a metric on the *tropical projective torus* $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. Moreover, we have

$$
\text{tdist}(\lambda \cdot v, \lambda \cdot w) = |\lambda| \cdot \text{tdist}(v, w) \quad \text{for all } \lambda \in \mathbb{R} \ .
\tag{2.2}
$$

This distance function is valid for both, min and max, as the tropical addition, denoted by $\oplus$ in the sequel.

Now let $A = (a_{ij}) \in \mathbb{R}^{d \times d}$ be a square matrix. We write $a_{i\cdot}$ for the ith row and $a_{\cdot k}$ for the kth column of A. The *tropical diameter* of A is the maximum

$$
\text{tdiam } A \ := \ \max_{i,j \in [d]} \text{tdist}(a_{i\cdot}, a_{j\cdot}) \ = \ \max_{i,j,k,\ell} \left| a_{ik} - a_{i\ell} + a_{j\ell} - a_{jk} \right| \ .
$$

Notice that the tropical diameter is the same as the diameter of a complete metric graph on d nodes with non-negative edge lengths. Further, the tropical diameters of a square matrix and its transpose agree.

Example 1. The tropical diameter of the ordinary $d \times d$-unit matrix, with ones on the diagonal and zeros elsewhere, equals two.

Observe that the tropical diameter does not change if the rows or columns of A are permuted.

Definition 2. *We now define the* tropical volume *of A as the expression*

$$
\text{tvol } A \ := \ \left| \bigoplus_{\sigma \in \text{Sym}(d)} \sum a_{i,\sigma(i)} - \bigoplus_{\tau \in (\text{Sym}(d) - \sigma_{opt})} \sum a_{i,\tau(i)} \right| \ ,
$$

where σ_{opt} is an optimal solution of the first optimization in the above.

In other words, σ_{opt} is a permutation for which $\sum a_{i,\sigma_{\text{opt}}(i)}$ coincides with the tropical determinant of A. Like the tropical diameter also the tropical volume is insensitive to transposing the matrix A or to any reordering of its rows or columns. The tropical volume can be computed in $O(d^3)$ time [6, §5.4.1].

Unlike the tropical diameter, which is an established notion, our definition of the tropical volume is new, at least under this name. Since our results below rely on this notion in a crucial way, a few words are in order. The classical volume has its foundation in measure theory, and the classical determinant yields the (normalized) volume

of a simplex. Tropical polytopes arise by "dequantization" of classical polytopes, or, if one prefers, as images of ordinary convex polytopes over real Puiseux series under the valuation map; this was first observed by Develin and Yu [10], and this is the point of departure of [3]. This leads to a notion of dequantized tropical volume with several good properties, but we defer the discussion to Section 4.

We prefer our definition of the tropical volume, tvol, since it leads to more interesting isoperimetric problems. The fact that it captures an essential metric property of tropical polytopes can be seen from the following observations. First, the tropical volume is a higher-dimensional generalization of the tropical distance function: indeed, in the linear case $d = 2$ the tropical diameter and the tropical volume agree, i.e., $\operatorname{tdiam} A = \operatorname{tvol} A$ if A is a $2{\times}2$-matrix. More importantly, the tropical volume provides a measure of non-singularity: it vanishes if and only if the rows (or the columns) of A are contained in a tropical hyperplane [22, Lemma 5.1]. In terms of statistical physics, the tropical volume is an energy gap, which appeared in the analysis of a non-standard optimal assignment algorithm by Kosowsky and Yuille. Their key result [20, Theorem 9] estimates the speed of convergence by an increasing function of the energy gap. Characterizing matrices with a maximal energy gap, knowing bounds on their entries, is precisely a tropical isoperimetric problem.

We call two square matrices *equivalent* if they can be transformed into one other by row and column permutations or by operations as in (2.1). Up to reordering the rows and columns we may assume that the identity permutation attains the tropical determinant. Since neither the tropical diameter nor the tropical volume changes if we translate each column by the same vector, we may assume that the first column is the vector $(1, 0, 0, \ldots, 0)$. Further, we can (ordinarily) add any multiple of $\mathbf{1}$ to each column without changing the tropical diameter, the tropical determinant or the tropical volume. Thus we may assume that each column, except for the first, begins with a zero. We call a matrix max-*standard* if the identity is an optimal permutation and if the first row and column read $(1, 0, 0, \ldots, 0)$. Each square matrix is equivalent to a max-standard matrix. In view of the Example 1 we will subsequently normalize the tropical diameter to two.

Theorem 3 (Tropical isodiametric inequality). *Assume that $\oplus = \max$ is the tropical addition. Let $A \in \mathbb{R}^{d \times d}$ be a matrix with tropical diameter two. Then the tropical volume does not exceed two. Moreover, if $\operatorname{tvol} A = 2$ then A is equivalent to a* max-*standard matrix whose coefficients a_{ij} satisfy the following conditions:*
   (i) $-1 \leqslant a_{ij} \leqslant 1$,
  (ii) $a_{ii} = 1$,
 (iii) $a_{ji} = -a_{ij}$ for $i \neq j$, and
 (iv) $-1 \leqslant a_{ij} + a_{jk} + a_{ki} \leqslant 1$ for i, j, k distinct.
Conversely, the tropical diameter and the tropical volume of each standard matrix satisfying these four conditions both equal two.

We call a matrix min-*standard* if the identity is an optimal permutation and if the first row and column read $(0, 1, 1, \ldots, 1)$. Each square matrix is equivalent to a min-standard matrix.

Corollary 4 (Tropical isodiametric inequality)**.** *Assume that* $\oplus = \min$ *is the tropical addition. Let* $B \in \mathbb{R}^{d \times d}$ *be a matrix with tropical diameter and tropical volume two. Then* B *is equivalent to a* min-*standard matrix whose coefficients* b_{ij} *satisfy the following conditions:*

   (i) $0 \leqslant b_{ij} \leqslant 2$,

  (ii) $b_{ii} = 0$,

 (iii) $b_{ij} + b_{ji} = 2$ *for* $i \neq j$, *and*

 (iv) $2 \leqslant b_{ij} + b_{jk} + b_{ki} \leqslant 4$ *for* i, j, k *distinct.*

Conversely, the tropical diameter and the tropical volume of each standard matrix satisfying these four conditions both equal two.

In the sequel we will be concerned with non-negative $d \times d$-matrices B which satisfy the conditions (ii), (iii) and (iv) in Corollary 4. We call any matrix which is equivalent to such a matrix *tropically near-isodiametric* (with respect to min). The matrix is *tropically isodiametric* if additionally the tropical diameter and the tropical volume are equal to two.

Proposition 5. *Let* $B \in \mathbb{R}^{d \times d}$ *be a tropically near-isodiametric matrix with respect to* $\oplus = \min$ *as the tropical addition. Then the matrix equation* $B \odot_{\min} B = B$ *holds.*

Notice that here we do *not* assume B to be standard. That is, we do not specify the first row and column. Also we do not require the upper bound in property (i), such that and the coefficients may be larger than two.

The *conical tropical convex hull* of a $d \times m$-matrix M, denoted as tpos M, is the set $\{M \odot x \mid x \in \mathbb{R}^m\}$. Since this is a homogeneous notion we usually consider tpos M as a subset of the tropical projective torus $\mathbb{R}^d / \mathbb{R}\mathbf{1}$. The following statement is phrased without an explicit reference to a tropical addition. It works in both cases.

Corollary 6. *Let* $M \in \mathbb{R}^{d \times d}$ *be near-isodiametric. Then the conical tropical convex hull of the columns (or the rows) of* M *is convex in the ordinary sense, i.e., it is a polytrope.*

Note that, even for general B, the Kleene star B^* is the shortest path matrix for the digraph on d nodes whose weights are given by the coefficients of B; see also [18, §3.4]. A polytrope is *isodiametric* if it arises from a tropically isodiametric matrix via Corollary 6.

Example 7. For $d = 3$ any isodiametric min-standard matrix looks like

$$B(\lambda) \;=\; \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \lambda \\ 1 & 2 - \lambda & 0 \end{pmatrix} \;,$$

where $0 \leqslant \lambda \leqslant 2$. The planar polytropes which arise as the min-tropical convex hulls of the columns of the matrices $B(\lambda)$ are shown in Figure 1 for various values of λ. The red points mark the non-redundant generators, i.e., the columns, while the white points are the pseudo-vertices, generically. Going from min to max means to interchange the roles of the red and the white points. For the non-generic cases $\lambda = 0$ and $\lambda = 2$ the non-redundant generators for min and max agree.
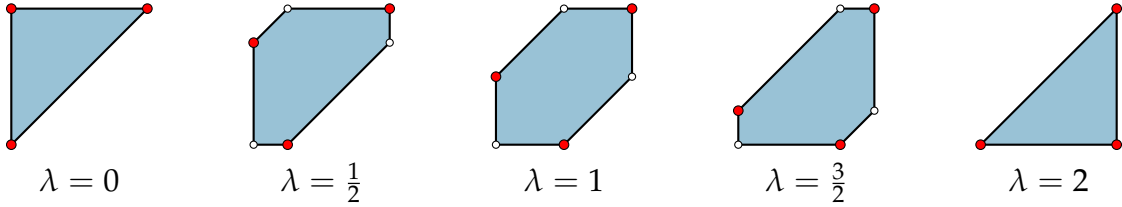

$$\lambda = 0 \qquad \lambda = \tfrac{1}{2} \qquad \lambda = 1 \qquad \lambda = \tfrac{3}{2} \qquad \lambda = 2$$

Figure 1: Isodiametric polytropes in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

As shown in Example 7 the isodiametric polytropes (with fixed diameter) depend on one real parameter which is, moreover, bounded between zero and two. In the general case we have $((d-1)^2 - (d-1))/2 = (d^2 - 3d)/2 + 1$ free parameters which are constrained by linear inequalities. That is, the isodiametric polytropes in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ are parameterized by a convex polytope $\mathrm{Iso}(d)$ of that dimension. While this is naturally embedded in a real vector space of dimension d^2, we usually look at its faithful projection into the coordinate directions given by the coefficients b_{ij} for $2 \leqslant i < j \leqslant d-1$. Notice that, up to this projection, $\mathrm{Iso}(d)$ is contained in the dilate $2 \cdot [0,1]^{d-1}$ of the unit cube by a factor of two. The polytope $\mathrm{Iso}(3)$ is the segment $[0,2]$.

3   Combinatorics of near-isodiametric polytropes

Let $B = (b_{ij})$ be an $d \times d$-matrix which is tropically near-isodiametric with respect to min. We want to analyze the polytrope $P := \mathrm{tpos}(B)$ seen as an ordinary convex polytope in $\mathbb{R}^{d-1}$. That latter space is identified with $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ via the map $(x_1, x_2, \ldots, x_d) \mapsto (x_2 - x_1, \ldots, x_d - x_1)$. Our point of departure is the exterior description

$$P(B) \; = \; \{x \in \mathbb{R}^d \mid x_i - x_j \leqslant b_{ij} \text{ for } i \neq j\} \tag{3.1}$$

as a *weighted digraph polyhedron*; see [21, §5.2] and [18]. Since $B = B^*$ is a Kleene star all these inequalities are tight; this classical result follows, e.g., from [13, §2.3.3]. The one-dimensional lineality space of $P(B)$ is $\mathbb{R}\mathbf{1}$. Let us set $P'(B) := P(B)/\mathbb{R}\mathbf{1}$.

Proposition 8. *Assume that the inequalities (iv) in Corollary 4 are strict, i.e.,*

$$2 < b_{ij} + b_{jk} + b_{ki} < 4 \quad \text{for all } i, j, k \; .$$

Then $P'(B)$ is an ordinary polytope of dimension $d-1$ with exactly $d(d-1)$ facets.

Any real $d \times n$-matrix M induces a height function on the vertices of the product of simplices $\Delta_{d-1} \times \Delta_{n-1}$. The induced regular subdivision is dual to (the covector decomposition of) the conical tropical convex hull of the columns of M; see [21, §5.2] and [18]. In the *generic* case that subdivision is a triangulation. Our main result says that there are isodiametric matrices which are generic.

Theorem 9. *For each $n \geqslant 3$ there exist isodiametric matrices B such that the polytrope $P'(B)$ is a simple ordinary polytope with exactly $\binom{2d}{d}$ vertices. In this case, the regular subdivision of $\Delta_{d-1} \times \Delta_{d-1}$ induced by B is a triangulation.*

Example 10. The matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 5/4 & 3/4 \\ 1 & 3/4 & 0 & 5/4 \\ 1 & 5/4 & 3/4 & 0 \end{pmatrix}$$

is a tropically isodiametric matrix, which is standard with respect to min, and which is generic. The resulting polytrope $P'(B)$, shown in Figure 2, is combinatorially equivalent to the second example in [17, Figure 2]. It belongs to Class 1 in the classification [15, §3.9]: among the 12 ordinary facets there are three quadrangles, six pentagons and three hexagons, and there is no pair of adjacent hexagons.
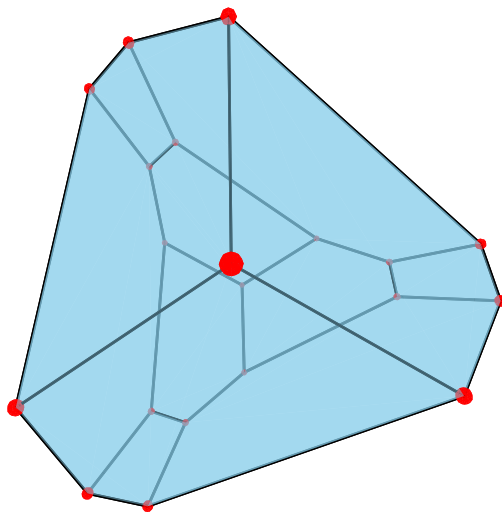

Figure 2: Isodiametric polytrope in $\mathbb{R}^4/\mathbb{R}1$

Remark 11. A computation with `polymake` [14] reveals the following: None of the other four combinatorial types of maximal polytropes for $n = 4$ from [17, Figure 2] admits a tropically isodiametric realization, nor does the sixth type, which was found by Jiménez and de la Puente [15, Example 21].

4 Tropical volume arising from dequantization

We now investigate a different notion of "volume", which is also applicable to tropical geometry. It arises from "dequantization", a term coined by Maslov for a procedure in which tropical objects are obtained as the log-limits of classical objects; see [19]. A related procedure is known as Viro's method [27] in real algebraic geometry.

In this approach, tropical numbers can be thought of as the images of the elements of a non-archimedean field under the valuation map. Following [2, 3], we will work with a field $\mathbb{K}$ of real valued functions in a real parameter t. We assume that $\mathbb{K}$ is a Hardy field whose elements are germs at infinity of real valued functions of t that are definable in a fixed polynomially bounded o-minimal structure. Then, the *valuation* of a function $f \in \mathbb{K}$ is defined by $\operatorname{val} f := \lim_{t \to \infty} (\log t)^{-1} \log |f(t)|$. The map val is a *non-archimedean* valuation, meaning that $\operatorname{val} 0 = -\infty$, $\operatorname{val} fg = \operatorname{val} f + \operatorname{val} g$, and $\operatorname{val} f + g \leqslant \max(\operatorname{val} f, \operatorname{val} g)$. Moreover, the latter inequality becomes an equality if $f, g \in \mathbb{K}_{\geqslant 0}$, the subset of non-negative functions of $\mathbb{K}$. We will assume that every function t^r with $r \in \mathbb{R}$ belongs to $\mathbb{K}$. Then, val yields a surjective morphism of semifields from $\mathbb{K}_{\geqslant 0}$ to the tropical semifield $\mathbb{T}$ (with ground set $\mathbb{R} \cup \{-\infty\}$ and maximum as addition).

The notions of convex hull, polyhedra, etc., make sense over $\mathbb{K}$. In particular, if $\mathcal{A} = (a_{ij}) \in \mathbb{K}^{d \times m}$, we denote by $\mathcal{P} := \operatorname{conv} \mathcal{A}$ the polytope generated by the columns of $\mathcal{A}$. By evaluating the matrix $\mathcal{A}(t) = (a_{ij}(t))$ at a real parameter t, we obtain a polytope $\mathcal{P}_t := \operatorname{conv} \mathcal{A}(t)$, so $\mathcal{A}$ encodes a parametric family of ordinary polytopes. We will denote by $\bar{\mathcal{A}}$ the $(d+1) \times m$ matrix obtained by adding an identically one row to the $d \times m$ matrix $\mathcal{A}$, putting this new row at the top of the matrix. If $\mathcal{A}$ has $m = d + 1$ affinely independent columns, $\mathcal{P}$ is a simplex with volume $(d!)^{-1} |\det \bar{\mathcal{A}}|$. In general, the volume of $\mathcal{P}$ can be computed by triangulating the configuration of points given by columns of $\mathcal{A}$ into simplices, just like over $\mathbb{R}$.

If $A = (a_{ij}) \in \mathbb{T}^{d \times m}$, we say that $\mathcal{A} = (a_{ij}) \in \mathbb{K}_{\geqslant 0}^{d \times n}$ is a *lift* of A if $\operatorname{val} \mathcal{A} = A$. While we worked with tropical cones and projective coordinates in Section 2, now it is more convenient to consider affine notions as follows. We call *affine tropical convex hull* of the columns of A, denoted as $\operatorname{tconv} A$, the set of vectors of the form $A \odot x$ where $x \in \mathbb{T}^m$ is such that $\max_i x_i = 0$. We will now use the notation $\bar{A}$ to denote the $(d+1) \times m$ matrix obtained by adding an identically zero row to A (still on the top row). Note also that if $A = \operatorname{val} \mathcal{A}$, we have $\operatorname{val} \bar{\mathcal{A}} = \bar{A}$. In this way, when the entries of A are finite, $\operatorname{tconv} A$ can be identified to the cross section by the hyperplane $x_1 = 0$ of the conical tropical convex hull, $\operatorname{tpos} \bar{A}$, defined in Section 2.

A result of Develin and Yu [10] implies that every affine tropical polytope P is of the form $\operatorname{val} \mathcal{P}$, where $\mathcal{P} = \operatorname{conv} \mathcal{A}$ for some matrix $\mathcal{A}$ with entries in $\mathbb{K}$. This motivates the following definition of tropical volumes of a polytope in terms of log-limits.

Definition 12. *Let $A \in \mathbb{T}^{d \times m}$. The* upper *and* lower *dequantized tropical volumes of A*

are defined by $\text{qtvol}^+ A := \sup\{\text{val} \text{vol} \mathcal{A} \mid \text{val} \mathcal{A} = A\}$ *and* $\text{qtvol}^- A := \inf\{\text{val} \text{vol} \mathcal{A} \mid \text{val} \mathcal{A} = A\}$, *respectively.*

Given a square matrix $C = (c_{ij})$ with entries in $\mathbb{T}$, we denote by $\text{tdet} \, C$ the tropical determinant of C (i.e., the value of the optimal assignment problem with weights c_{ij}). We have the following characterization of the upper dequantized tropical volume.

Theorem 13. *If $A \in \mathbb{T}^{d \times m}$, then*

$$\text{qtvol}^+ A = \max_{I \subset [m], |I| = d} \text{tdet} \, A[I] \ ,$$

where $A[I]$ denotes the maximal submatrix of A obtained by selecting the columns in I.

We shall say that a matrix A is *tropically sign-generic* if, in the optimal assignment problem associated to any maximal square submatrix of A, all the optimal permutations have the same parity.

Theorem-Definition 14 (Dequantized tropical volume). *If $A \in \mathbb{T}^{d \times m}$ is such that $\bar{A}$ is tropically sign generic, then, $\text{qtvol}^+ A = \text{qtvol}^- A$, and we denote by $\text{qtvol} \, A$, the dequantized tropical volume of A, this common value. Moreover, for any lift $\mathcal{A}$ of A, we have*

$$\lim_{t \to \infty} \frac{\log \text{vol} \, \text{conv} \, \mathcal{A}(t)}{\log t} = \text{qtvol} \, A \ . \tag{4.1}$$

The next proposition shows that $\text{qtvol} \, A$ depends only of the affine tropical convex hull of A.

Proposition 15. *Let $A \in \mathbb{T}^{d \times m}$. Suppose that A or $\bar{A}$ is tropically sign generic, that $B \in \mathbb{T}^{d \times p}$, and that $\text{tconv}(A) \subset \text{tconv}(B)$. Then $\text{qtvol}^+(A) \leqslant \text{qtvol}^+(B)$. In particular, if $\text{tconv}(A) = \text{tconv}(B)$ and if $\bar{A}, \bar{B}$ are both tropically sign generic, then $\text{qtvol}(A) = \text{qtvol}(B)$.*

This proposition will allow us to define, for those tropical polytopes that can be written as $P = \text{tconv}(A)$ with A tropically sign generic, the dequantized volume $\text{qtvol}(P) := \text{qtvol}(A)$.

Example 16. The tropical genericity condition cannot be dispensed with. Consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \ .$$

We have $\text{tconv} \, A \subset \text{tconv} \, B$. However, $\text{qtvol}^+ A = 0$, whereas $\text{qtvol}^+ B = -1$.

Corollary 17. *Let $A \in \mathbb{T}^{d \times m}$, $P := \text{tconv} \, A$, $B \in \mathbb{T}^{d \times p}$, $Q := \text{tconv} \, B$, $C := (A, B)$ and suppose that $\bar{C}$ is tropically sign generic. Then*

$$\text{qtvol}\big(\text{tconv}(P \cup Q)\big) = \max\big(\text{qtvol}(P), \text{qtvol}(Q)\big) \ .$$

In other words, the dequantized tropical volume is an *idempotent measure* [19, 1]. Dyer and Frieze [12] showed that computing the volume of a classical polytope given by its vertices is ♯P hard. This is in contrast with the tropical situation.

Corollary 18. *Let $A = (a_{ij}) \in \mathbb{T}^{d \times m}$. The upper dequantized tropical volume* $\mathrm{qtvol}^+ A$ *can be computed in strongly polynomial time.*

The dequantized tropical volume can be used to bound the volume of ordinary polytopes. Instead of considering the non-archimedean valuation val over $\mathbb{K}$, we shall consider the archimedean valuation $\log |\cdot|$ over $\mathbb{R}$. Given a matrix $A = (a_{ij}) \in \mathbb{R}_{\geqslant 0}^{d \times m}$, we denote by $\mathrm{Log}\, A$ the matrix obtained by applying the archimedean valuation entrywise.

Theorem 19. *Let $A = (a_{ij}) \in \mathbb{R}_{\geqslant 0}^{d \times m}$. Then*

$$\mathrm{vol\,conv}\, A \ \leqslant\ \alpha(d+1)\exp(\mathrm{qtvol}^+(\mathrm{Log}\, A))\ , \tag{4.2}$$

where α is the number of maximal cells of an arbitrary triangulation of the point configuration given by the columns of A.

The size of any triangulation of A, and thus also α, is bounded from above by $O(m^{\lceil (d+1)/2 \rceil})$; see [9, Cor. 2.6.2].

It is instructive to compare the dequantized tropical volume $\mathrm{qtvol}^\pm$ with the tropical volume tvol. When $A \in \mathbb{T}^{(n-1) \times n}$, the quantities $\mathrm{tvol}\, \bar{A}$ and $\mathrm{qtvol}^\pm A$ provide different "measures" of the singularity of the matrix $\bar{A}$. Indeed, one can check that $\mathrm{qtvol}^- A = \mathrm{qtvol}^+ A$ if and only if $\bar{A}$ is tropically sign generic, meaning that all maximizing permutations in $\mathrm{tdet}\, \bar{A}$ have the same sign, whereas $\mathrm{tvol}\, \bar{A} > 0$ if and only there is only one maximizing permutation. Therefore, $\mathrm{tvol}\, \bar{A} > 0$ implies that $\mathrm{qtvol}^+ A = \mathrm{qtvol}^- A$, but not vice versa. The dequantized tropical volume has several properties to be expected from a measure on tropical polytopes (like being defined for the convex hull of any number of points in general position and being an idempotent measure). However, the isoperimetric inequality for the dequantized volume may read $\mathrm{qtvol}^+ A \leqslant (n-1) \times \max_{ij} a_{ij}$. This leads to more degenerate isoperimetric results, since the matrices achieving the equality do not have such a rigid structure as the maximizing matrices in Theorem 3.

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